On the probability of generating a primitive matrix

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Joint work with Yong Feng, Yang Liu and Wenyuan Wu

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What is a primitive matrix?

Primitive vector $\boldsymbol{x} \in \mathbb{Z}^n$:

- Definition: $\boldsymbol{x} = d\boldsymbol{y}$ for $\boldsymbol{y} \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ implies $d = \pm 1$.
- Reiner '56: $\boldsymbol{x} \in \mathbb{Z}^n$ is primitive $\Longleftrightarrow$ $\boldsymbol{x}$ can be extended to an $n \times n$ **unimodular matrix** over $\mathbb{Z}$.

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Primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k \times n}$ with $k \leq n$:

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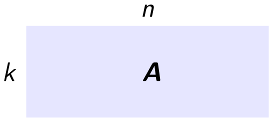
What is our problem?

- For a given primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k \times n}$ with $\|\boldsymbol{A}\| = \max_{i,j} |a_{i,j}| \leq \lambda$



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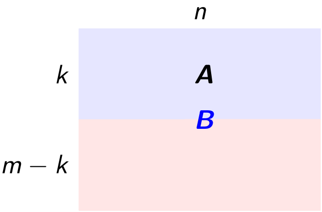

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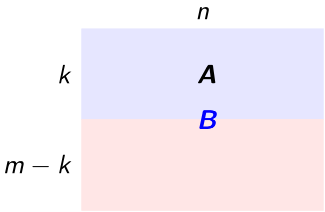

- **What is the probability** of that $\boldsymbol{B}$ is still primitive ?

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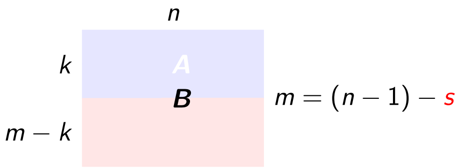
  - **Problem 3**: How fast is the algorithm ?

Related work on probability analysis



- Maze-Rosenthal-Wagner '11: For $k = 0$, $s \geq 0$, the natural density is

$$\prod_{j=s+2}^{n} \frac{1}{\zeta(j)} \quad (\lambda \to \infty),$$

where $\zeta(\cdot)$ is the Riemann's zeta function.

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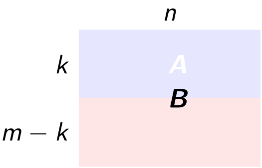

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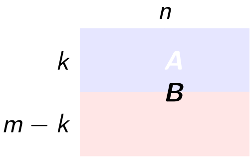
- Fontein-Wocjan '14:
  - For $k \geq 2n+1$, a probability is rigorously proven.
  - For $n+1 \leq k < 2n+1$, a probability is conjectured.
- ...

Our result on the probability

- A primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k\times n}$ with $\|\boldsymbol{A}\| \le \lambda$
- An integer s with $0 \le s \le n-k-2$
- $\boldsymbol{B} \in \mathbb{Z}^{(n-s-1)\times n}$: a completion of $\boldsymbol{A}$ with unif. rand. entries from Λ

Then the probability of that $\boldsymbol{B}$ is primitive is at least

$$1 - 4\left(\frac{2}{3}\right)^{s+1}\left(1-\left(\frac{2}{3}\right)^{n-k-s-1}\right) - \frac{2(n-s-1)^2}{\lambda^{s+2}}\left(1-\frac{1}{\lambda^{n-k-s-1}}\right).$$



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Roadmap

1. **Proof of the result**

2. **Application to unimodular matrix completion**

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The idea of the proof

For $i = k, \ \ldots, \ n - s - 1$, define

$$\boldsymbol{A}_i = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_i \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \end{pmatrix}.$$

Idea: Give an upper bound on the probability of the event that $\boldsymbol{A}_{n-s-1}$ is not primitive under the assumption that $\boldsymbol{A}_k$ is primitive.

Tool: If $\boldsymbol{A}_i$ is not primitive, then there must be at least one prime p such that $\operatorname{rank}(\boldsymbol{A}_i) \leq i - 1$ over $\mathbb{Z}_p$.

Some events and their probability

MDep_i: There exists at least one prime p s.t. $\mathrm{rank}(\boldsymbol{A}_i) \le i-1$ over $\mathbb{Z}_p$.

$\neg\mathsf{MDep}_i$: $\boldsymbol{A}_i$ is a primitive matrix.

Goal: Give an upper bound on $\Pr[\mathsf{MDep}_{n-s-1} | \neg\mathsf{MDep}_k]$.

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Dep$_i$: $\mathrm{rank}(\boldsymbol{A}_i) \leq i-1$ over $\mathbb{Q}$.

$$\Pr[\mathrm{MDep}_i | \neg\mathrm{MDep}_{i-1}] \leq \Pr[(\mathrm{MDep}_i \wedge \mathrm{Dep}_i) | \neg\mathrm{MDep}_{i-1}] + \Pr[(\mathrm{MDep}_i \wedge \neg\mathrm{Dep}_i) | \neg\mathrm{MDep}_{i-1}]$$

Bound $\Pr[\mathbf{MDep}_i | \neg \mathbf{MDep}_{i-1}]$

Let $\lambda \geq 2$ be an integer and $k+1 \leq i \leq n-3$.

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$$\Pr[(\mathsf{MDep}_i^{(p<\lambda)} \wedge \neg\mathsf{Dep}_i)|\neg\mathsf{MDep}_{i-1}] \leq \left(\frac{2}{3}\right)^{n-i+1} + \frac{3}{4}\left(\frac{1}{3}\right)^{n-i+1}.$$

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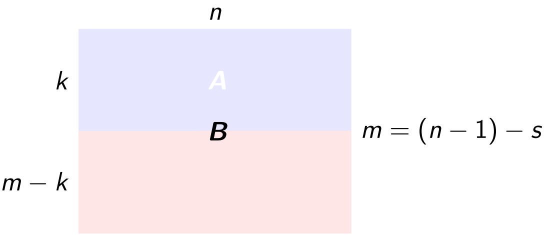

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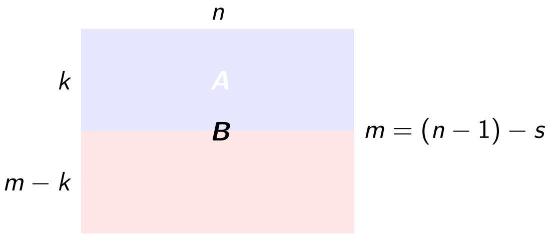

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A heuristic based on an extensively experimental study:

A constant lower bound on the probability exists for $s = 0,\ 1,\ 2$ as well.

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Hermite normal form

Non-singular matrix $\boldsymbol{H} \in \mathbb{Z}^{n \times n}$ is in Hermite normal form if

- $\boldsymbol{H}$ is upper triangular with non-negative entries,
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$$\begin{pmatrix} 1 & 0 & 0 & 10 & 260\,246\,748 \\ 0 & 1 & 0 & 2 & 292\,062\,707 \\ 0 & 0 & 1 & 7 & 244\,095\,302 \\ 0 & 0 & 0 & 14 & 342\,954\,195 \\ 0 & 0 & 0 & 0 & 344\,319\,363 \end{pmatrix}$$

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Algorithm 1

Input: An integer matrix $\boldsymbol{A} \in \mathbb{Z}^{n \times n}$.

Output: A matrix $\boldsymbol{B} \in \mathbb{Z}^{n \times n}$, with $\boldsymbol{B}$ equal to $\boldsymbol{A}$ except for the last column, $\|\boldsymbol{B}\| \leq n^2 \|\boldsymbol{A}\|$, and the last diagonal of HNF($\boldsymbol{B}$) equal to 1.

Proposition

Given an $n \times n$ integer matrix $\boldsymbol{A}$, Algorithm 1 is a correct Las Vegas algorithm and requires at most $O(n^{\omega+\varepsilon} \log^{1+\varepsilon} \|\boldsymbol{A}\|)$ bit operations.

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$$\boldsymbol{B} = \begin{pmatrix} -66 & -65 & 20 & -90 & -14 \\ 55 & 5 & -7 & -21 & 2 \\ 68 & 66 & 16 & -56 & 17 \\ 13 & -41 & -62 & -50 & 4 \\ 26 & -36 & -34 & -8 & -4 \end{pmatrix} \quad \boldsymbol{BP} = \begin{pmatrix} -14 & -66 & -65 & 20 & -90 \\ 2 & 55 & 5 & -7 & -21 \\ 17 & 68 & 66 & 16 & -56 \\ 4 & 13 & -41 & -62 & -50 \\ -4 & 26 & -36 & -34 & -8 \end{pmatrix}$$

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Unimodular matrix completion

Theorem

Given a primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k\times n}$, there exists a Las Vegas algorithm that completes $\boldsymbol{A}$ to an $n \times n$ unimodular matrix $\boldsymbol{U}$ such that

$$\|\boldsymbol{U}\| \leq n^{O(1)}\|\boldsymbol{A}\|$$

in an expected number of

$$O(n^{\omega+\varepsilon}\log^{1+\varepsilon}\|\boldsymbol{A}\|)$$

bit operations.

- The standard method: $O((n-k)n^{\omega+\varepsilon}\log^{1+\varepsilon}\|\boldsymbol{A}\|)$.

Conclusion

Given a primitive $\boldsymbol{A} \in \mathbb{Z}^{k \times n}$, consider to complete $\boldsymbol{A}$ to an $(n-s-1) \times n$ matrix with uniformly random integers in $[0, \|\boldsymbol{A}\|)$.

- We present a rigorous proof of the probability for $3 \leq s \leq n-k-2$.
    - Previously, only the limit probability when $\lambda \to \infty$ is known for $k = 0$.

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THANKS