Encrypted matrix operations via bicyclic encoding

陈经纬

中国科学院重庆绿色智能技术研究院
ChongQing Institute of Green and Intelligent Technology,Chinese Academy of Sciences

Joint work with Linhan Yang, Wenyuan Wu, Yang Liu, and Yong Feng

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Roadmap

1. **Homomorphic encryption**
   - Basics
2. **Plaintext matrix–ciphertext vector multiplication**
   - Fast linear transformation by Halevi-Shoup (2014, 2015)
3. **Ciphertext matrix–ciphertext matrix multiplication**
   - HE-friendly expression by Jiang et al. (2018)
4. **Bicyclic encoding and matrix multiplication**
   - Ciphertext – ciphertext matrix multiplication
   - Performance

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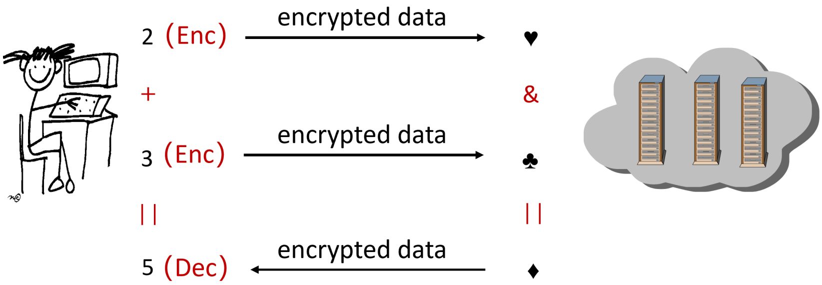
Homomorphic encryption (HE)



- ✓ Communication security
- ✓ Storage security
- ✓ Computation security
- ✓ Post-quantum security

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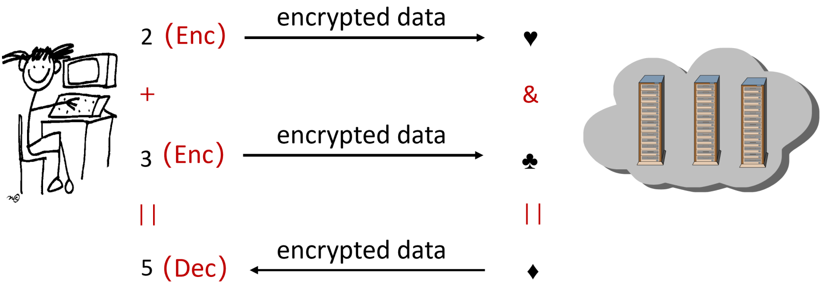

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Applications

- Secure outsourcing
- Non-interactive computation
- Privacy-preserving ML

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  - Proof of concept implementations
- 3^{rd} and 4^{th} generation, and more (2015-present)
  - Fast bootstrapping: DM, CGGI, ...
  - Real number arithmetic: CKKS
  - Further optimizations:
    - $\rightsquigarrow$ LW '23, XZDDF '23, MHWW '24, WWLLWLW '24, ...
  - Public libraries: HElib, SEAL, OpenFHE, TFHE, Lattigo, ...
  - Practical applications: iDASH competition, Standardization, ...

Functionality of SIMD-supported HE schemes

- Packing method
  - Enc: $\boldsymbol{x} \mapsto \mathtt{ct} = \mathtt{Enc}(\boldsymbol{x})$, where $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$

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    - ⇝ $\circ$: Hadamard product, i.e., element-wise multiplication
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  - Cost: no multiplicative depth but a key switching
- Compositions of the basic operations
  - TSum: $\mathtt{Enc}(x_0, \ldots, x_{n-1}) \mapsto \mathtt{Enc}(s, \ldots, s)$ with $s = x_0 + \cdots + x_{n-1}$
  - Cost of TSum: $\#\mathtt{Rot} = \log n$

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Naïve method

Linear transformation on a ciphertext

- A plain $n \times n$ matrix $\boldsymbol{A}$
- A ciphertext $\mathtt{ct} = \mathtt{Enc}(\boldsymbol{x})$ of an n-dimensional vector $\boldsymbol{x}$
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A naïve algorithm

For $i = 0, 1, \ldots, n-1$ do the following

- Compute $\texttt{ct}_i = \texttt{CMul}(\boldsymbol{a}_i, \texttt{ct})$ // $\boldsymbol{a}_i$: the i-th row of $\boldsymbol{A}$; $\texttt{ct}_i = \texttt{Enc}(\boldsymbol{a}_i \circ \boldsymbol{x})$
- Compute $\texttt{ct}_i = \texttt{TSum}(\texttt{ct}_i)$ // $\texttt{ct}_i = \texttt{Enc}(\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle, \ldots, \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle)$

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- Cost of the naïve algorithm
  - #resulting ciphertexts: n
  - #CMul: n
  - #Rot: $n \log n$

Diagonal vectors of a matrix and linear transformation*

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} \cdot \begin{pmatrix}a\\ b\\ c\end{pmatrix} = \underbrace{\begin{pmatrix}1\\ 5\\ 9\end{pmatrix}}_{\boldsymbol{d}_0} \circ \underbrace{\begin{pmatrix}a\\ b\\ c\end{pmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{pmatrix}2\\ 6\\ 7\end{pmatrix}}_{\boldsymbol{d}_1} \circ \underbrace{\begin{pmatrix}b\\ c\\ a\end{pmatrix}}_{\mathrm{Rot}_1(\boldsymbol{x})} + \underbrace{\begin{pmatrix}3\\ 4\\ 8\end{pmatrix}}_{\boldsymbol{d}_2} \circ \underbrace{\begin{pmatrix}c\\ a\\ b\end{pmatrix}}_{\mathrm{Rot}_2(\boldsymbol{x})} = \begin{pmatrix}1a+2b+3c\\ 4a+5b+6c\\ 7a+8b+9c\end{pmatrix}$$

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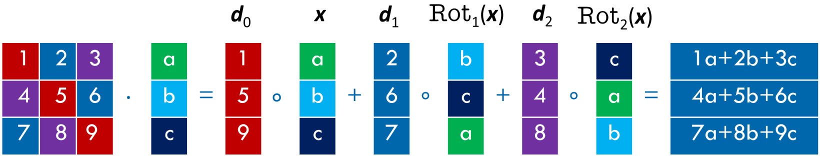
Diagonal vectors of a matrix and linear transformation*



- Diagonal vectors of an $n \times n$ matrix $\boldsymbol{A}$: $\boldsymbol{d}_0, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_{n-1}$
- Now we have

$$\boldsymbol{A}\boldsymbol{x} = \sum_{0 \le i < n} \boldsymbol{d}_i \circ \mathtt{Rot}_i(\boldsymbol{x})$$

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- Cost of the fast linear transformation on a ciphertext
  - #resulting ciphertexts: 1
  - #`CMul`: n
  - #`Rot`: n

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The baby-step-giant-step (BSGS) strategy[†]

- Reducing $n = \ell \cdot k$ rotations to $\ell + k$

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- Cost of the even faster linear transformation on a ciphertext
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Some sparse linear transformations[‡]

- Assume only $r = \ell \cdot k$ non-zero diag. vectors with indices satisfying
  - $g(ki + j) = g(ki) + g(j)$ for all $0 \leq i < \ell$ and $0 \leq j < k$
- Then we have

$$\boldsymbol{Ax} = \sum_{0 \leq i < \ell} \texttt{Rot}_{g(ki)} \left(\sum_{0 \leq j < k} \texttt{Rot}_{-g(ki)} \left(\boldsymbol{d}_{g(ki+j)} \right) \circ \texttt{Rot}_{g(j)}(\boldsymbol{x}) \right)$$

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- Cost of such a linear transformation
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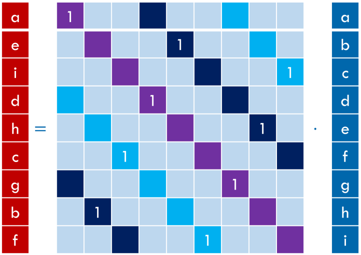
- Cost of such a linear transformation
  - #resulting ciphertexts: 1
  - #`CMul`: r
  - #`Rot`: $2\sqrt{r}$
- E.g., useful for permutation



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Matrix encoding

- Identify a $d \times d$ matrix as a vector of dimension $n = d^2$ (row by row)
- Cost of some basic matrix operations
  - Matrix addition: $\texttt{\#Add} = 1$
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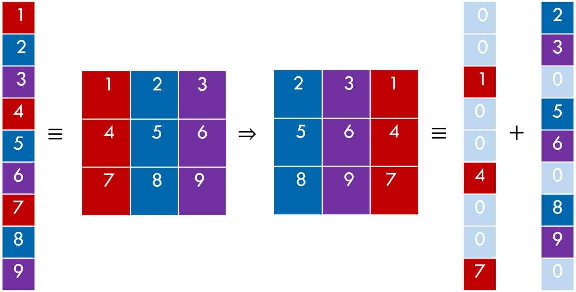
  - Rotation among columns: `#Rot` $= 2$, `#CMul` $= 2$



Matrix multiplication (d, d, d)[‡]

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

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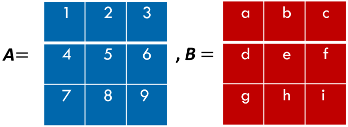
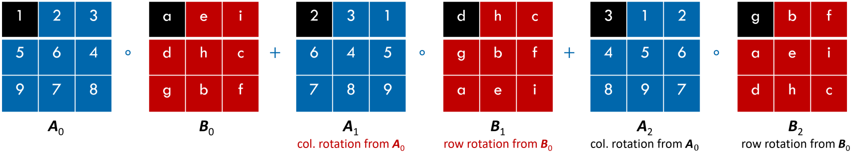


- For $d \times d$ **SQUARE** matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A}_0 \circ \boldsymbol{B}_0 + \cdots + \boldsymbol{A}_{d-1} \circ \boldsymbol{B}_{d-1}$$



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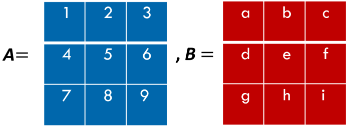

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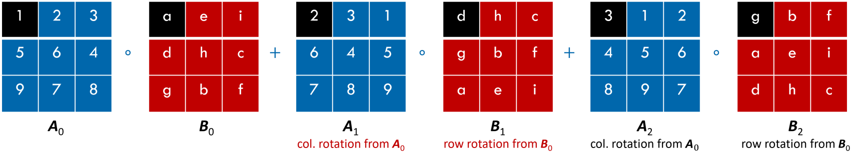
- Cost of a square matrix multiplication
  - Generation of $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$
  - `#Mul`: d; `#Add`: $d-1$



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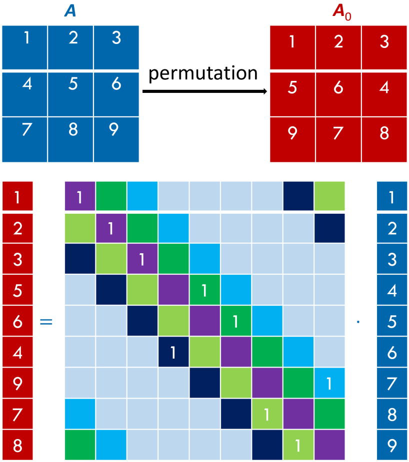

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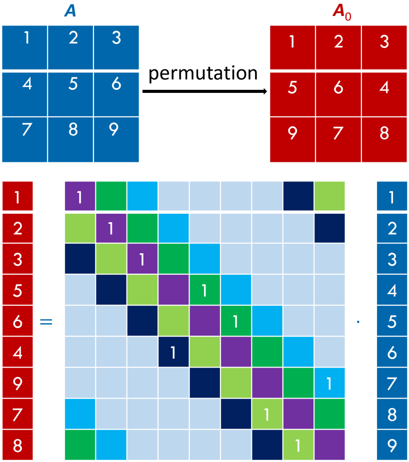

- Cost for $\boldsymbol{A}_0$ (permutation)
  - #Non-zero diag. vectors: $2d-1$
  - #CMul: $2d-1$
  - #Rot: $2\sqrt{2d-1} \leq 3\sqrt{d}$
- Cost for $\boldsymbol{A}_i$ from $\boldsymbol{A}_0$ (column rotation)
  - #CMul: $2d$
  - #Rot: $2d$
- Cost for $\boldsymbol{B}_0$ (permutation)
  - #Non-zero diag. vectors: d
  - #CMul: d
  - #Rot: $2\sqrt{d}$
- Cost for $\boldsymbol{B}_i$ from $\boldsymbol{B}_0$ (row rotation)
  - #Rot: d

Ciphertext matrix–ciphertext matrix multiplication (d, d, d)

Method	#Ctxts	#Mul	#CMul	#Rot	Mult. depth
Naïve	d^2	d^2	0	$d^2 \log d$	$1\,\texttt{Mul}$
Halevi-Shoup	d	d^2	0	$2d\sqrt{d}$	$1\,\texttt{Mul}$
JKLS	1	d	$5d$	$3d + 5\sqrt{d}$	$1\,\texttt{Mul} + 2\,\texttt{CMul}$

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- Can we do better?

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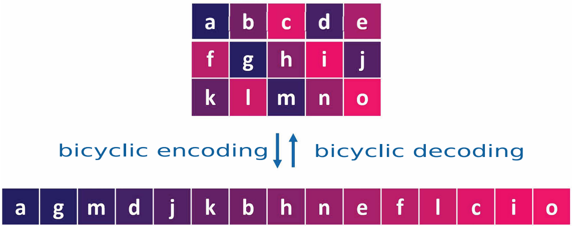

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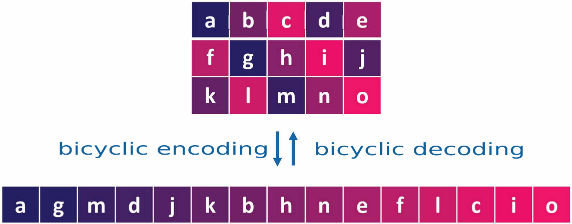
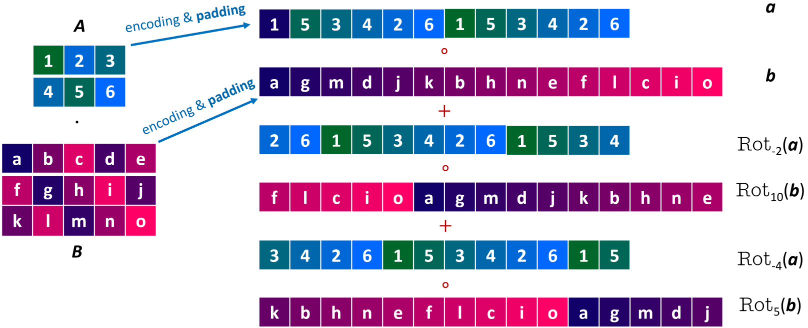
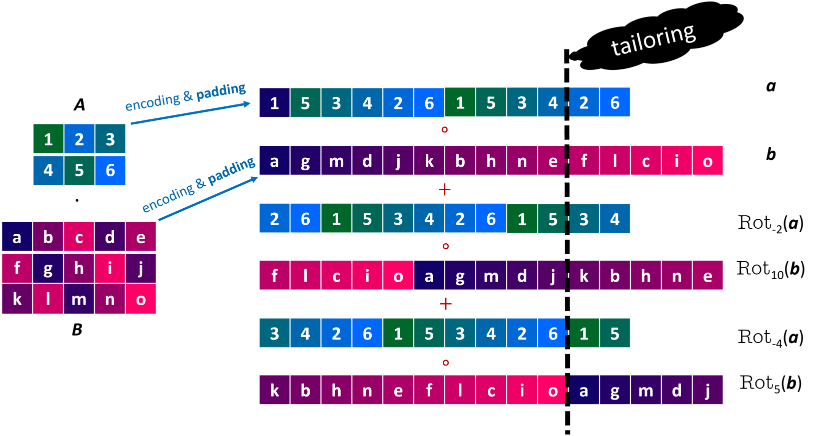


- Basic idea: $\mathbb{Z}/(mn)\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times m$ and $m \times p$ matrices with n, m, p coprime.



Matrix multiplication $(n, m, p) = (2, 3, 5)$



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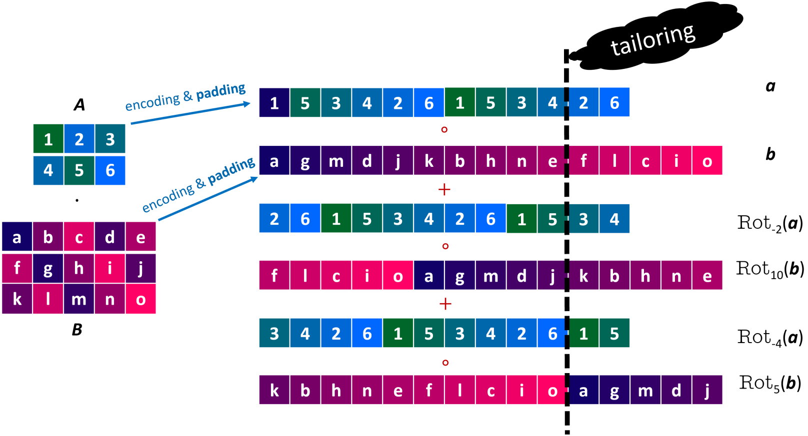

- Cost (n, m, p are coprime)
  - #Mul: m
  - #Rot: $2m + \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{p}{m} \right\rceil$
  - #CMul: 1
  - Mult. depth: $1\,\mathtt{Mul} + 1\,\mathtt{CMul}$

Encrypted matrix multiplication (n, m, p)

Method	#Ctxts	#Mul	#CMul	#Rot	Mult. depth
Naïve	np	np	0	$np \log m$	1 Mul
Halevi-Shoup*	p	pd	0	$2p\sqrt{d}$	1 Mul
JKLS†	1	d	$5d$	$3d + 5\sqrt{d}$	1 Mul + 2 CMul
BMM-I‡	1	m	0	$2m + 2$	1 Mul

*$d = \max(n, m)$
†$d = \max(n, m, p)$
‡$n,\ m,\ p$ coprime, $\max(n, p) < m$
§$n,\ m,\ p$ coprime, m is a power-of-2 integer

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BMM-II†§	1	1	0	$3 \log d$	1 Mul

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‡ n, m, p coprime, $\max(n, p) < m$

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Higher dimension: block versus segment

Given #slots ℓ, a high dim. matrix will be encrypted into **many** ciphertexts.

- Block matrix multiplication
  - E.g., Strassen algorithm $O(d^{\log_2 7})$
    - ⇝ Recursive: Large memory cost
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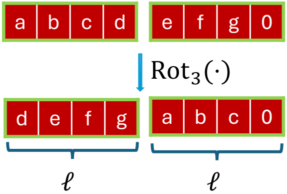

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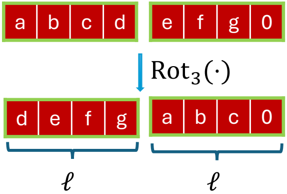
- **BMM-III**: Segment matrix multiplication
  - Depends on `Segment` version operations
  - #`Mul`: $m \cdot \lceil np/\ell \rceil$
  - #`CMul`: $(4\lceil np/\ell \rceil + 2)m + n + p$
  - #`Rot`: $2m \cdot \lceil np/\ell \rceil$
  - Mult. depth: 1 `CMul` + 1 `Mul`



Roadmap

1. **Homomorphic encryption**
   - Basics
2. **Plaintext matrix–ciphertext vector multiplication**
   - Fast linear transformation by Halevi-Shoup (2014, 2015)
3. **Ciphertext matrix–ciphertext matrix multiplication**
   - HE-friendly expression by Jiang et al. (2018)
4. **Bicyclic encoding and matrix multiplication**
   - Ciphertext – ciphertext matrix multiplication
   - Performance

Implementation and setup

- BMM-I, BMM-II
- Strassen block version of BMM-I, BMM-II
- BMM-III
- Based on CKKS in Microsoft SEAL
  - Ciphertext space: $\mathbb{Z}[X]/\langle X^N + 1, q\rangle$
  - $\log q \approx 50 + L \cdot \log \Delta + 60$
  - L: the number of multiplicative depth
- Matrix entries: $\texttt{pow}(-1, \texttt{i} + \texttt{j}) * \texttt{rand}()/\texttt{pow}(2, 30)$
- Security: ≥ 128 bits
- Error: $\leq 10^{-2}$

Small square matrix multiplication

Table: Performance comparison with the R.-T. algorithm for small-dimensional matrices. $N = 8192$ and $\log \Delta = 30$.

Method	$\log q$	Dimension	Time (ms)	Speedup
R.-T.*	170	$(16, 16, 16)$	199	1.0x
BMM-I	170	$(16, 19, 17)$	130	1.5x
BMM-I	140	$(16, 19, 17)$	82	2.4x
BMM-II	140	$(15, 16, 17)$	**13**	16.6x

*P. Rizomiliotis and A. Triakosia, On matrix multiplication with homomorphic encryption, in Proc. 2022 on Cloud Computing Security Workshop

Square matrix multiplication

Table: Performance comparison for $(128, 128, 128)$ matrix multiplication. $\log \Delta = 30$ except for BMM-III with $\log \Delta = 40$.

Method	$\log q$	$N = 8192$		$N = 32768$	
		Basic block	Time (s)	Basic block	Time (s)
Naïve block JKLS *	200	$(64, 64, 64)$	11.34	$(128, 128, 128)$	14.17
Strassen + JKLS *	200	$(64, 64, 64)$	10.34	$(128, 128, 128)$	14.17
Naïve block BMM-I	140	$(43, 45, 44)$	7.59	$(86, 89, 87)$	13.32
Strassen + BMM-I	140	$(32, 35, 33)$	8.31	$(64, 67, 65)$	11.99
Naïve block BMM-II	140	$(15, 16, 17)$	**4.40**	$(21, 32, 23)$	8.32
Strassen + BMM-II	140	$(11, 8, 9)$	84.89	$(17, 16, 19)$	127.69
BMM-III	190	$(128, 131, 129)$	11.05	$(128, 131, 129)$	41.60

*X. Jiang, M. Kim, K. Lauter, and Y. Song, Secure outsourced matrix computation and application to neural networks, CCS '18

Square matrix multiplication

Table: Performance comparison for $(256, 256, 256)$ matrix multiplication, $\log \Delta = 40$.

Method	$\log q$	$N = 8192$		$N = 32768$	
		Basic block	Time (s)	Basic block	Time (s)
Naïve block JKLS *	200	$(64, 64, 64)$	89.33	$(128, 128, 128)$	112.01
Strassen JKLS *	200	$(64, 64, 64)$	71.54	$(128, 128, 128)$	97.90
Naïve block BMM-I	140	$(43, 45, 44)$	59.57	$(86, 89, 87)$	73.35
Strassen + BMM-I	140	$(32, 35, 33)$	56.98	$(64, 67, 65)$	80.99
Naïve block BMM-I	140	$(15, 16, 17)$	76.60	$(21, 32, 23)$	76.19
BMM-III	190	$(256, 259, 257)$	**42.90**	$(256, 259, 257)$	110.99

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Square matrix multiplication

Table: Performance comparison for matrix multiplication of dimension 512 and 1024 with $N = 8192$.

Method	$\log q$	(512, 512, 512) Basic block	(512, 512, 512) Time (s)	(1024, 1024, 1024) Basic block	(1024, 1024, 1024) Time (s)
Naïve block JKLS *	200	(64, 64, 64)	728	(64, 64, 64)	6028
Strassen JKLS *	200	(64, 64, 64)	479	(64, 64, 64)	3514
Naïve block BMM-I	140	(43, 45, 44)	490	(43, 45, 44)	4240
Strassen + BMM-I	140	(32, 35, 33)	390	(32, 35, 33)	2757 [†]
Naïve block BMM-II	140	(15, 16, 17)	1766	(15, 16, 17)	–
BMM-III	190	(512, 515, 513)	**181** [†]	(1024, 1027, 1025)	**1200** [†]

[†] By default, $\log \Delta = 30$ except for these with $\log \Delta = 40$.

*X. Jiang, M. Kim, K. Lauter, and Y. Song, Secure outsourced matrix computation and application to neural networks, CCS '18

Rectangular matrix multiplication

Table: Performance for rectangular matrix multiplication (I).

Huang et al.'s algorithm *†		BMM-III‡			
Dimension	Time (s)	Dimension	KeyGen (s)	Total (s)	Speedup
$(256, 256, 16)$	**6.19**	$(256, 257, 17)$	16.15	23.60	
$(256, 16, 256)$	**6.23**	$(256, 17, 257)$	14.69	19.37	
$(1024, 1024, 16)$	108.22	$(1024, 1025, 17)$	14.68	**55.79**	1.9x
$(1024, 16, 1024)$	108.31	$(1024, 17, 1025)$	14.47	**43.64**	2.4x
$(2048, 2048, 8)$	218.09	$(2048, 2049, 11)$	14.49	**98.01**	2.2x
$(2048, 8, 2048)$	218.09	$(2049, 8, 2051)$	14.63	**81.09**	2.6x

† For Huang et al.'s algorithm, N is set as the same as theirs and $\log \Delta = 30$.

‡ For BMM-III, $N = 8192$ and $\log \Delta = 40$.

*Z. Huang, C. Hong, C. Weng, W.-j. Lu, and H. Qu, More efficient secure matrix multiplication for unbalanced recommender systems, IEEE TDSC, 2023

Rectangular matrix multiplication

Table: Performance (in sec.) for rectangular matrix multiplication (II). $N = 8192$, $\log \Delta = 30$.

Dimension	Naïve block JKLS *	Naïve block BMM-I	Speedup
$(4, 1636, 5)$	36.92	**9.76**	3.7x
$(8, 3405, 9)$	78.58	**19.92**	3.9x
$(16, 6903, 17)$	157.00	**38.57**	4.0x
$(32, 13847, 33)$	317.31	**76.73**	4.1x

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Rectangular matrix multiplication

Table: Performance (in sec.) for rectangular matrix multiplication (III). $N = 8192$, $\log\Delta = 30$.

Dimension	Naïve block JKLS *	Naïve block BMM-I	LKS* + Segment + Opt.
$(4, 5, 1636)$	36.31	**0.10**	0.25
$(8, 9, 3405)$	77.56	0.65	**0.63**
$(16, 17, 6903)$	157.32	4.96	**2.23**
$(32, 33, 13847)$	316.76	38.82	**8.24**

*W. Lu, S. Kawasaki, and J. Sakuma, Using fully homomorphic encryption for statistical analysis of categorical, ordinal and numerical data, NDSS '17

Conclusion

- Encrypted matrix multiplication (n, m, p pairwise coprime)
- More flexible and more efficient
- Paper: https://doi.org/10.1109/TIFS.2024.3490862
- Code: https://github.com/hangenba/bicyclic_mat_mul
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Open problems

- How to remove the coprime condition?
- Encrypted BLAS? Fast vector/matrix/tensor ops on encrypted data